ON STRAIN THEORIES OF PLASTICITY FOR SINGULAR LOADING SURFACES

PMM Vol. 31, No. 5, 1967, pp. 887-889

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(Recived March 11, 1967)

Budianskii has shown in [1] that, when load surfaces are singular, then the strain relationships need not contradict the basic concepts of the theory of plasticity.

Kliushnikov developed in [2] his relations of the theory of flow of a plastic body and obtained explicit formulas for strain, applicable to a fairly wide selection of the loading processes.

Ravotnov proposed in [3] a two-dimensional model of a strain-hardening plastic body and illustrated the peculiarities occurring in relations of the theory of plastic flow when nodes are formed on the local surfaces. He showed that relations of the Hencky-Nadai strain theory can also be found when the processes of loading are not proportional. A number of distinctive characteristics of the behavior of the load function in the two-dimensional case was also discussed in [4].

We note that several years earlier, Hodge [5] integrated the formulas of the theory of flow for the cases when the states of stress corresponded to the singularities of a linear approximation for the local surface.

Below we study the conditions under which the relations of the theory of flow lead, for piecewise smooth load surfaces, to Hencky-Nadai strain relations.

1. Relations of the theory of plastic flow when load surfaces are piecewise smooth, can be written as [6]

$$\varepsilon_{ij}^p = \sum_k h_k \frac{\partial f_k}{\partial \sigma_{ij}} \left(\frac{\partial f_k}{\partial \sigma_{ij}} d\sigma_{ij} \right), \qquad f_k(\sigma_{ij}, e_{ij}^p, \chi_i, k_i) = 0 \qquad (1.1)$$

$$h_k = h_k(\sigma_{ij}, e_{ij}^p, \chi_i, k_i)$$

where ϵ_{ij}^p, e_{ij}^p and σ_{ij} are components of the rate of plastic strain, plastic strain and stress, respectively; χ_i are nonholonomic parameters dependent on the process of loading; k_i are constants and h_k are strain-hardening functions.

In the following we shall, for simplicity, consider the case of rigid-plastic body and shall, consequently, drop the superscript p.

Expression (1.1) contains first derivatives of f_k, therefore we can replace f_k, in the six-dimensional space of the symmetric stress tensor σ_{ij}, by any set of functions $g_i = 0$ such, that when values of the parameters σ_{ij}, e_{ij} and χ_i are given, the $\partial f_k / \partial \sigma_{ij}$ and $\partial g_k / \partial \sigma_{ij}$ coincide completely.

The above requirement can be defined as a property resulting from a localized character of relations of the theory of plastic flow; instantaneous character of deformability is determined by the values of the first derivatives $\partial f_k / \partial \sigma_{ij}$ for given σ_{ij}, e_{ij}, χ_i, k_i and h_k and does not depend on other characteristics of the load surface.

In the following we shall assume that the load functions are independent of χ_i.

In order to show clearly our basic reasonong we shall assume in the beginning, that the stress and strain state is defined by just two pairs of stress and strain components different from zero (e. g. the case of torsion or of a skew-plane strain)

$$\sigma_{12}, \sigma_{13}, e_{12}, e_{13} \neq 0 \tag{1.2}$$

Then the load function will have the form

$$f_k(\sigma_{12}, \sigma_{13}, e_{12}, e_{13}) = 0 \tag{1.3}$$

In the stress space functions $f_k = 0$ are represented by the corresponding surfaces, while the magnitudes e_{ij} serve as parameters. Any two independent functions of (1.3) can be considered as some finite relations which, in general, will yield explicit formulas

$$e_{12} = \varphi_{12}(\sigma_{12}, \sigma_{13}), \qquad e_{13} = \varphi_{13}(\sigma_{12}, \sigma_{13}) \tag{1.4}$$

which define the required strain relations. Expressions of the associated law of flow(1.1) yield, in this case, the hardening functions h_k.

Let us now assume that out of each set of stress and strain components, three are different from zero (a plane problem)

$$\sigma_{11}, \sigma_{22}, \quad \sigma_{12}, \quad e_{11}, \quad e_{22}, \quad e_{12} \neq 0 \tag{1.5}$$

If there exist three independent functions $f_k = 0$ then we can, generally speaking, find

$$e_{ij} = \varphi_{ij}(\sigma_{11}, \sigma_{22}, \sigma_{12}), \qquad i, j = 1, 2 \tag{1.6}$$

If in the three-dimensional subspace σ_{ij} a singularity of the local function corresponds to a conical point, then that point can be considered as an envelope of tangent planes. Only three out of the family of planes with a common point at the apex of the cone are independent in the three-dimensional space. Other planes can be obtained as linear conbination of the independent ones. Hence, three independent relations $g_k(\sigma_{ij}, e_{ij}) = 0$ are sufficient to determine the strain relations (1.6). The general case can be approached in a similar manner. Strain relations

$$e_{ij} = \varphi_{ij}(\sigma_{mn}) \tag{1.7}$$

will occur, if six independent finite relations

$$g_k(\sigma_{ij}, e_{ij}) = 0 \tag{1.8}$$

exist for a given singularity on the load surface.

2. Relations of the theory of small elastoplastic strains have the form [7]

$$e_{ij}' = \frac{e_u}{\sigma_u}\sigma_{ij}', \qquad e_u = \Phi(\sigma_u), \qquad \sigma = Ke$$
$$\sigma_u = (\sigma_{ij}'\sigma_{ij}')^{1/2}, \quad e_u = (e_{ij}e_{ij}')^{1/2}, \quad \sigma = {}^1/_3\sigma_{ii}, \quad e = {}^1/_3 e_{ii} \tag{2.1}$$
$$\sigma_{ij}' = \sigma_{ij} - \delta_{ij}\sigma, \qquad e_{ij}' = e_{ij} - \delta_{ij}e$$

Let us introduce six independent variables

$$X = \sigma - Ke, \qquad X_{ij} \equiv e_{ij} - (1/K)\sigma - (\Phi(\sigma_u)/\sigma_u)\sigma_{ij}' \tag{2.2}$$

Magnitudes X_{ij} are, obviously, components of a certain tensor. If the load functions have the form

$$f_k(X, X_{ij}) = 0 \tag{2.3}$$

and the roots $(X, X_{ij}) = 0$ represent a solution of the set of six Eqs. $g_k(X, X_{ij}) = 0$, then the relations (2.1) obviously hold. Load functions (2.3) should be tensor invariant.

Let us form a set of independent invariants

$$X = \sigma - Ke, \quad X_{ij}\sigma_{ij}, \quad X_{ij}e_{ij}, \quad X_{ij}\sigma_{jk}\sigma_{ki}, \quad X_{ij}e_{jk}e_{ik}, \quad X_{ij}\sigma_{jk}e_{ki} \tag{2.4}$$

Load functions $f_k = 0$ dependent on (2.4) are tensor invariant. If the solutions of the set $g_k = 0$ consists of all the invariants of (2.4) equated to zero, then the arbitrariness of σ_{ij} and e_{ij} implies that $X_{ij} = 0$, i.e. we have (2.1). Obviously, when the relations of the theory of small elastoplastic strains are proved in this manner, then the requirements of incompessibility $K = \infty$, exponential dependence $\sigma_u = A e_u$ and of proportionality of load, are no longer necessary unlike the case of smooth load surfaces [7].

3. As an example we shall consider such load functions, that functions $g_k = 0$ can be defined in the form

$$X + Y = 0, \qquad X - Y = 0 \tag{3.1}$$

$$X = e_x - \psi\tau_x, \qquad Y = e_y - \psi\tau_y, \qquad \psi = \psi(\tau_x^2 + \tau_y^2)$$

from which the relations of the strain theory follow at once,

$$e_x = \psi\tau_x, \qquad e_y = \psi\tau_y \tag{3.2}$$

Let us now assume a load, under which only τ_x and e_x are different from zero. Let us find the singularities of the load function near the point $M(\tau_x, 0)$. Relations (3.1) have in this case the form

$$e_x - \psi(\tau_x + \tau_y) = 0, \qquad e_x - \psi(\tau_x - \tau_y) = 0, \qquad e_y = 0 \tag{3.3}$$

Differentiating (3.3) we obtain, at the point M,

$$\frac{d\tau_y}{d\tau_x} = \pm\left(\frac{2\psi_0'}{\psi_0}\tau_x^2 + 1\right) \tag{3.4}$$

where the prime accompanying ψ denotes differentiation with respect to its argument $\tau_x^2 + \tau_y^2$, while the subscript 0 denotes the fact that the relevant values correspond to the point $M(\tau_x, 0)$.

Relation (3.4) gives the value of the angle of the load surface at the point $M(\tau_x, 0)$.

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Translated by L. K.